J. Appl. Maths Mechs Vol. 57, No. 2, pp. 295–299, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

THE MOTION OF A SOLID IN A FLOW OF PARTICLES†

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(*Received 27 January* 1992)

The problem of the motion of a solid in a flow of particles around a fixed point is considered. This problem is well known to have an extremely non-conservative form. Nevertheless, it turns out that the dynamics of the solid in this problem can be described, with certain assumptions, by a system of Hamiltonian equations. The conditions under which this quite unexpected fact can occur, are investigated.

The presence of a Hamiltonian structure considerably increases the interest in the problem of the existence of additional first integrals of the equations of motion. It turns out that there are certain cases when these integrals exist. These include the case when the equations of motion allow the possibility of an integral similar to a Hess integral in the problem of the motion of a heavy solid around a fixed point. The steady-state motions of the system considered are also determined, and their stability is investigated.

THE STUDY of the forces and moments of the forces of the interaction between a solid and a medium go back to Newton [1]. His investigations were continued by numerous others (see, for example, [2]). The present state of the investigation of the interaction between a body and a medium, and also an investigation of different dynamic effects which occur even for the simplest models of the interactions, is described in [3].

1. We will consider the problem of the motion of a solid in a flow of gas in the following formulation. Suppose the gas consists of identical non-interacting particles, moving with constant velocity in a fixed direction in a stationary absolute space. Suppose the particles interact absolutely inelastically with the solid, i.e. after collision the velocity of a particle with respect to the solid is zero. Suppose the surface of the solid is convex. Then, if the stream velocity considerably exceeds the product of the characteristic value of the angular velocity of the solid and the characteristic scattering from the solid to a fixed point, the equations of motion of the solid can be represented in the form‡

$$I\omega^{\cdot} = I\omega \times \omega + f\gamma \times c(\gamma)S(\gamma), \quad \gamma^{\cdot} = \gamma \times \omega \tag{1.1}$$

where $I = \operatorname{diag}(I_1, I_2, I_3)$ is the inertia tensor of the solid with respect to the fixed point O, $Ox_1x_2x_3$ is a system of coordinates whose axes are directed along the principal axes of inertia of the inertia tensor at the point O, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular-velocity vector, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$

†*Prikl. Mat. Mekh.* Vol. 57, No. 2, pp. 77–81, 1993.

‡Compare with the result of the paper by V. V. Sazonov, A mechanism of the loss of stability of the gravitational orientation of an artificial satellite. Preprint No. 107. Institute of Applied Mathematics, Academy of Sciences of the U.S.S.R., Moscow, 1988.

is the unit vector directed along the stream, and f is a constant, the value of which is proportional to the density of the gas and the square of the stream velocity.

In Eq. (1.1) $S(\gamma)$ is the area of the shadow $T(\gamma)$ of the solid on the plane $\pi(\gamma)$, perpendicular to the stream velocity, i.e. the projection of the solid on the π plane along the γ direction, and $c(\gamma)=(c_1(\gamma)\ c_2(\gamma),\ c_3(\gamma))$ is the vector connecting the fixed point with any point on the straight line $l(\gamma)$ parallel to γ and containing the centroid of the shadow—the point coinciding with the centre of mass of a uniform plate occupying the region T.

Equations (1.1) have an integral invariant of unit density, and also first integrals $J_1=(I\omega,\ \gamma)$ and $J_2=\gamma^2$. Equations (1.1) are reversible, i.e. they can be subjected to a replacement of variables and time $(\omega,\ \gamma,\ t)\to(-\omega,\ \gamma,\ -t)$. However, in general, these equations are not a system of Hamilton equations with some Poisson structure.The following assertion holds.

Assertion 1. If, for any i, j, $i\neq j$, the following relations are satisfied

$$c_i\frac{\partial S}{\partial\gamma_j}+\frac{\partial c_i}{\partial\gamma_j}S(\gamma)=c_j\frac{\partial S}{\partial\gamma_i}+\frac{\partial c_j}{\partial\gamma_i}S(\gamma) \tag{1.2}$$

the equations of motion are Hamiltonian with a Poisson structure, defined by an $E(3)$ algebra, and can have an additional first integral.

Proof. Suppose

$$c(\gamma)S(\gamma)=\partial U/\partial\gamma \tag{1.3}$$

for a certain function $U(\gamma)$. Then, if the function U is sufficiently smooth, to satisfy relations (1.3) it is necessary and sufficient for conditions (1.2) to be satisfied. The equations of motion can then be represented in the form

$$\begin{gathered} M^{\cdot}=\{M,H\},\quad \gamma^{\cdot}=\{\gamma,H\} \\ \{M_i,M_j\}=\varepsilon_{ijk}M_k,\quad \{M_i,\gamma_j\}=\varepsilon_{ijk}\gamma_k,\quad \{\gamma_i,\gamma_j\}=0 \end{gathered} \tag{1.4}$$

where the Hamilton function

$$H=\tfrac{1}{2}(I^{-1}M,M)+U(\gamma) \tag{1.5}$$

defines $J_0=H$—an additional first integral of Eqs (1.1)—the analogue of the energy integral.

We will point out some cases when relations (1.2) are satisfied and Eqs (1.1) possess a Hamiltonian structure.

(*a*) *The surface of the solid is centrally symmetric*. In this case, the vector $c(\gamma)=c$ connects the fixed point and the centre of symmetry. Equations (1.2) can then be represented in the form

$$c_i\partial S/\partial\gamma_j=c_j\partial S/\partial\gamma_i$$

and have the general solution

$$S=S((c,\gamma)) \tag{1.6}$$

The potential can now be represented in the form

$$U(\gamma)=f\int_0^{(c,\gamma)}S(u)du$$

Hence, if the expression for the area of the shadow is described by a relation of the form (1.6), the equations of motion of the solid will be Hamiltonian.

Cases exist when relations (1.6) are satisfied.

The surface of a solid is spherical. The area of the shadow is then constant and the equations of motion are identical with the equations of motion of a solid in a uniform force field.

The surface of the solid which interacts with the particle flux is a centrally symmetric plate. In this case, $S = S_0 |c|^{-1}(c, \gamma)$, where S_0 is the area of the plate, and the interaction potential has the form

$$U(\gamma) = \tfrac{1}{2} f S_0 \operatorname{sign}(c,\gamma)(c,\gamma)^2 |c|^{-1}$$

However, relations (1.6) are not always satisfied. If the surface of the solid is an ellipsoid, whose axes are collinear with the principal axes of inertia of the solid, we have

$$S = \pi b_1 b_2 b_3 (\gamma_1^2 / b_1^2 + \gamma_2^2 / b_2^2 + \gamma_3^2 / b_3^2)^{1/2}$$

where b_1, b_2 and b_3 are the semi-axes of the ellipsoid. In this case, relation (1.6) will not, in general, be satisfied.

(*b*) *The surface of the solid is axisymmetric.* In this case, the straight line $l(\gamma)$ intersects the axis of symmetry of the surface of the solid. Then, if the fixed point is situated on the axis of symmetry, we have $c(\gamma) = \chi((\alpha, \gamma))(\alpha_1, \alpha_2, \alpha_3)$, $S = S((\alpha, \gamma))$ and the potential U can be represented in the form

$$U(\gamma) = f \int_0^{(\alpha,\gamma)} S(u)\chi(u)du$$

An ellipsoid of rotation. Suppose the semi-axes of the ellipsoid b_1 and b_2 are equal to b, $c = (0, 0, c_3)$. Then

$$S(\gamma) = \pi b^2 b_3 \left(\frac{1}{b^2} + \left(\frac{1}{b_3^2} - \frac{1}{b^2} \right) \gamma_3^2 \right)^{1/2}$$

and the potential has the form

$$U(\gamma_3) = \pi b b_3 c_3 f \int_0^{\gamma_3} \left(1 + \frac{b^2 - b_3^2}{b_3^2} u^2 \right)^{1/2} du$$

2. We will point out some cases when the equations of motion have an additional integral.

A trivial case. Suppose the surface of the solid is centrally symmetric and the centre of symmetry coincides with the point of suspension. Equations (1.1) then admit of an integral $J_3 = (I\omega)^2$. In this case, the problem is completely integrable and is identical with the Euler–Poinsot problem.

The case of axial symmetry. Suppose the body is dynamically symmetrical, that is, for example, the condition $I_1 = I_2$ is satisfied. Suppose also that the surface of the solid is centrally symmetric and the centre of symmetry lies on the Ox_3 axis. The equations of motion will then have a first integral $J_3 = \omega_3$. This case is similar to the Lagrange case.

Analogues of the Hess case. 1. Suppose the surface of the body is centrally symmetric, and the centre of symmetry and the moments of inertia are such that

$$I_1 < I_2 < I_3, \ (I_1^{-1} - I_2^{-1})^{1/2} c_3 \mp (I_2^{-1} - I_3^{-1})^{1/2} c_1 = 0, \ c_2 = 0$$

Then, the equations of motion have a partial integral

$$F = (I_1^{-1} - I_2^{-1})^{1/2} I_1\omega_1 \pm (I_2^{-1} - I_3^{-1})^{1/2} I_3\omega_3 = 0$$

2. Suppose the surface of the solid is axisymmetric, and the axis of symmetry is defined by the vector α and contains the point of suspension. Then, if the relation, which differs from (2.1) by having c_1, c_2 and c_3 replaced by α_1, α_2 and α_3, is satisfied, the equations of motion will have a partial integral (2.1). These two cases are similar to the Hess case.

3. The equations of motion (1.1) have partial solutions, for which the solid undergoes rotation with constant angular velocity ω around the streamlines of the gas. In this case, the angular velocity of the solid and the components of the vector γ are related by the equation

$$\omega^2 I\gamma \times \gamma + f\gamma \times c(\gamma)S(\gamma) = 0 \tag{3.1}$$

The set of axes of rotation forms in the space $R^3(\gamma)$ a conical surface consisting of straight lines passing through the origin of coordinates and the points on the sphere $\gamma^2 = 1$ such that

$$(I\gamma, \gamma \times c(\gamma)) = 0$$

Suppose $c(\gamma) = (0, 0, c_3(\gamma))$. Then Eqs (3.1) have the solutions $\gamma^{\pm} = (0, 0, \pm 1)$. Suppose $c(\gamma)$ and $S(\gamma)$ are fairly smooth functions. Then the necessary conditions for these solutions to be stable have the form

$$\begin{aligned} &I_1 I_2 \Omega^2 - I_1 c_3 f S\gamma_3 - I_2 c_3 f S\gamma_3 - (I_3 - I_2)(I_1 - I_3)\Omega^2 \geqslant \\ &\geqslant 2(I_1 I_2((I_1 - I_3)(I_2 - I_3)\Omega^4 + (I_1 + I_2 - 2I_3)\Omega^2\gamma_3 c_3 f S + c_3^2 f^2 S^2))^{1/2} \end{aligned} \tag{3.2}$$

$$(I_1 - I_3)(I_2 - I_3)\Omega^4 + (I_1 + I_2 - 2I_3)\Omega^2\gamma_3 c_3 f S + c_3^2 f^2 S^2 \geqslant 0 \tag{3.3}$$

If $I_1 = I_2$, condition (3.3) is always satisfied, while condition (3.2) can be represented in the form

$$I_3^2\omega^2 \geqslant 4I_1 f S c_3\gamma_3 \tag{3.4}$$

In the case of strict inequality, condition (3.4) is then not only necessary but also sufficient.

4. Suppose the surface of the solid possesses a plane of symmetry coinciding with one of the coordinate planes of the system of coordinates $Ox_1x_2x_3$, for example, with the Ox_1x_2 plane. Then the system of equations (1.1) has a partial solution, for which

$$\gamma_3 \equiv 0, \quad \omega_1 = \omega_2 = 0$$

In this case, the equations of plane oscillations can be represented in the form of a Hamiltonian system. Suppose φ is an angle such that $\gamma_1 = \cos\varphi$, $\gamma_2 = \sin\varphi$, $\omega_3 = \varphi^{\bullet}$. Then

$$I_3\varphi^{\bullet\bullet} = f(\cos\varphi c_2(\cos\varphi, \sin\varphi, 0) - \sin\varphi c_1(\cos\varphi, \sin\varphi, 0))S(\cos\varphi, \sin\varphi, 0)$$

or

$$\frac{d\varphi}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial \varphi},\quad H=\tfrac{1}{2}I_3^{-1}p^2+U(\varphi)$$

$$U(\varphi)=-\int\limits_0^{\varphi} f(\cos\varphi c_2(\cos\varphi,\sin\varphi,0)-\sin\varphi c_1(\cos\varphi,\sin\varphi,0))$$
$$S(\cos\varphi,\sin\varphi,0)d\varphi$$

where the Hamilton function H is their first integral (compare with [4]).

Hence, even in the simplest mechanical model considered it is possible to observe fairly interesting dynamic properties. At the same time, the question of the effect of the neglected terms in the expression for the moments deserves a separate investigation.

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Translated by R.C.G.